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ON THE DYNAMIC SNAP-THROUGH OF A NON-LINEAR ELASTIC SYSTEM*

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The dynamic instability "in the large" of a non-linear elastic continuous conservative system with Rayleigh friction subjected to a load applied instantaneously at the same $t=0$ and keeping a constant value for all $t\geqslant 0$ /1-6/ is investigated. By using a potential theory analysis, the concepts of a well and equilibrium stability factor introduced by Myshkis /7, 8/, definitions are given of the dynamic stability of the system, the critical load of its dyanmic snap-through, and the astatic critical load. The latter yields a lower limit of those values of the load for which dynamic snap-through occurs. For the class of systems with a potential energy of the form of the square of the norm plus a weakly continuous functional /9/, among which are shallow elastic shells, for instance, /10-12/, it is proved that the existence of saddle points with negative index follows from the non-uniqueness of the stable equilibrium. Here at least one saddle point is found on the boundary of the well of each stable equilibrium. Therefore, the stability factor acquires a graphic meaning as the least crossing among the energetic peaks leading from the well of a given equilibrium to the wells of the other equilibria (or to infinity, which is impossible, it is true, for functionals increasing at infinity). The application of this property for systems with potential energy depending on the load parameter p is the fundametal effective calculation of the stability factor and the astatic critical load p_a.

A basis is presented for the applicability of the energetic approach for the non-linear vibrations equations of elastic shallow shells. In particular, the classical problem is examined of the dynamic snap-through of a shallow elastic spherical shell subjected to an instantaneously applied hydrostatic load, and an example is presented of the determination of the astatic ritical load p_a in the case of ambiguity of the families of unstable equilibria. We note that the load p_a for shallow spherical shells with different geometric parameters and boundary conditions has been found earlier in /13-17/. The good agreement between the values of p_a and the critical load of dynamic snap-through, obtained by a direct numerical integration of the non-stationary problem /2-6, 13-18/ indicates the efficiency of using the energetic approach developed here in the theory of shells.

The reasoning associated with estimating the height of the energetic barrier was used earlier in finite-dimensional models of the Galerkin method for the equations of the vibration of an arch /1, 19-21/. The considerations presented later understandably also include the case of systems with a finite number of degrees of freedom.

1. Equation of non-linear elastic system vibrations. Domain of possible motions.

Consider the vibrations equation of a conservative mechanical system in the presence of viscous friction forces in a separable Hilbert space H:

$$\omega_{tt}+\beta B\omega_t+I'(\omega)=0,\; I'(\omega)=\operatorname{grad}_H I(\omega)$$
$$I(\omega)={}^1/_2(A\omega,\,\omega)_H-\varphi(\omega) \tag{1.1}$$

Eq.(1.1) has been introduced in /11/ as the abstract model of a system of non-linear equations of motion of shallow elastic shells. Here $\omega(t)$ is an unknown vector-function of time t with values in $H, I(\omega)$ is the system potential energy, $\beta B\omega_t$ is the Rayleigh friction, and β is the coefficient of friction.

We will assume that the following conditions are satisfied.

1) The operators A, B are linear. Their domains of definition D_A, D_B are compact in H.

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The operator A is selfadjoint, positive-definite, and with a completely continuous inverse operator A^{-1}. We introduce the scalar product in the set D_A

$$(\omega_1, \omega_2)_{H_A} = (A\omega_1, \omega_2)_H \quad (1.2)$$

The closure of D_A in the norm (1.2) yields a complete Hilbert space H_A, the energetic space of the operator A /22/. The imbedding of H_A in H is completely continuous.

The operator B is a selfadjoint, positive-definite operator acting limitedly from H into H.

2) The functional $I(\omega)$ is given in the whole space H_A and grows in ω: $I(\omega) \to \infty$ as $\|\omega\|_{H_A} \to \infty$. Using (1.2) we rewrite $I(\omega)$ in the form:

$$I(\omega) = {}^1/_2 \|\omega\|^2_{H_A} - \varphi(\omega) \quad (1.3)$$

Here $\varphi(\omega)$ is a weakly continuous, twice continuously differentiable functional in H_A. Then $\varphi'(\omega)$ is a completely continuous operator acting from H_A into H (the theorem of E.S. Tsitlanadze /9/, p.85/.

Let us specify the initial conditions

$$\omega|_{t=0} = g_0, \quad \omega_t|_{t=0} = g_1; \quad g_0 \in H_A, \quad g_1 \in H \quad (1.4)$$

We obtain the energy dissipation equation from (1.1), (1.4)

$$E_t = -\beta(B\omega_t, \omega_t)_H \leqslant 0 \quad (1.5)$$
$$E(t) = {}^1/_2 \|\omega_t\|_H{}^2 + I(\omega)$$

Integrating with respect to time between s and t, and setting $s = 0$ and $s = \infty$, we arrive at the inequality

$$E(\infty) \leqslant E(t) \leqslant E(0)$$
$$E(0) = {}^1/_2 \|g_1\|_H{}^2 + I(g_0)$$

Let $\beta = 0$. Then system (1.1) is conservative and the law of conservation of energy is satisfied for it: $E(t) = E(0)$. It hence follows that for $\beta = 0$ the possible motions of system (1.1) are in a domain V defined by the inequality

$$I(\omega) \leqslant E(0) \quad (1.6)$$

2. Stable equilibrium wells and energy saddle points. We will establish certain properties of the functional $I(\omega)$. For any $j \in R$ we introduce a set of least values of the energy level j:

$$M_j = \{\omega \in H_A : I(\omega) \leqslant j\}$$

Lemma 2.1. The set M_j is bounded and weakly closed. The boundary ∂M_j is a closed, but generally not weakly closed set.

Proof. If it is assumed that M_j is not bounded, then there exists a sequence $\omega_n \in M_j$ such that $\|\omega_n\|_{H_A} \to \infty$. Since $I(\omega)$ is a function that grows in ω, we obtain $I(\omega_n) \to \infty$, but this contradicts the inequality $I(\omega_n) \leqslant j$. Therefore, M_j is a bounded set. Let $\omega_n \overset{st}{\to} \omega_0$ strongly, where $\|\omega_n\|_{H_A} \to d$. Then $\varphi(\omega_n) \to \varphi(\omega_0)$ and $\|\omega_0\|_{H_A} \leqslant d$. Passing to the limit in the inequality $I(\omega_n) \leqslant j$ as $n \to \infty$, because of the semicontinuity of the norm relative to weak convergence, we deduce ${}^1/_2 d^2 - \varphi(\omega_0) \leqslant j$. Applying both inequalities we obtain

$$I(\omega_0) = {}^1/_2 \|\omega_0\|^2_{H_A} - \varphi(\omega_0) \leqslant {}^1/_2 d^2 - \varphi(\omega_0) \leqslant j$$

i.e., M_j is a weakly closed set. The boundary $\partial M_j = \{\omega \in H_A : I(\omega) = j\}$ is a closed set. Nevertheless, ∂M_j is not absolutely a weakly-closed set (/9/, p.303).

The critical points (CP) of the functional I are determined from the equations

$$I'(\omega) \equiv \omega - \varphi'(\omega) = 0, \quad I'(\omega) = \mathrm{grad}_H\, I(\omega) \quad (2.1)$$

We let C_I denote the set of CP of the functional I: $C_I = \{\omega \in H_A : I'(\omega) = 0\}$.

Lemma 2.2. Let $u \in C_I$. Then the Fréchet differential $I''(u) = 1 - \varphi''(u)$ has no more than a finite number of negative eigenvalues.

Proof. The linear operator $\varphi''(u)$ is completely continuous (/9/, p.140). Its spectrum can have just one limit point $\lambda = 0$, hence, outside the neighbourhood of the point $\mu = 1$ $(\mu = 1 - \lambda)$ the spectrum of the operator $1 - \varphi''(u)$ has just a finite number of negative eigenvalues.

Two corollaries result from Lemma 2.2.

Corollary 2.1. The CP of the function I cannot be a point of its relative maximum in H_A.

It is understood that this result only holds in the infinite-dimensional case. Maxima sometimes appear /1/ for systems with a finite number of degrees of freedom that approximate elastic systems, which explicitly indicates the inadequacy of the approximation.

Corollary 2.2. If the spectrum of the operator $1 - \varphi''(u)$ is positive, then $u \in C_I$ is a relative minimum point of the functional I in H_A.

Definition. The CP s of the functional $I(\omega)$ is called a saddle point if the Fréchet differential $1 - \varphi''(s)$ has at least one negative eigenvalue. The sum of the multiplicities of the negative eigenvalues of the operator is called the type of saddle point.

Then as is well-known (/9/, p.303), the functional $I(\omega)$ has at least one minimum point. Let $K_j(m)$ be a component of the connectedness of the set M_j that contains a minimum point m. Obviously values $j\,(\geqslant I(m))$ exist for which $K_j(m)$ does not contain other CP.

Lemma 2.3. If the boundary $\partial K_j(m)$ does not contain CP, then for $\omega \in \partial K_j(m)$ there exists a number $\alpha > 0$ such that $\| I'(\omega) \|_H \geqslant \alpha$.

The proof duplicates the well-known discussion (/9/, p.112).

Let $j^* = \sup j$, where the upper bound is taken over those energy values j for which $K_j(m)$ does not contain other CP except m. The notation $K_{j^*}(m) = K^*$ is henceforth used.

Theorem 2.1 (about the saddle point). Let m and n be isolated points of the relative minimum of the functional I. Then in the non-degenerate case the boundary ∂K^* contains at least one saddle point of the functional I: $\partial K^* \cap C_I = \varnothing$, ∂K^* does not contain maxima and minima.

Proof. For the relative minimum $I(m)$ there exists a neighbourhood $G(m) \in H_A$ such that $I(\omega) > I(m)\ \forall \omega \in G(m) \setminus m$. We examine the sequence of points $\omega^l \in K^*$ such that $I(m) < j_1 < j_2 < \ldots \to j^* = \sup I(\omega)$, $\omega \in K_1^*$, $j_l = K(\omega^l)$, where j_l is a monotonically increasing sequence of numbers for which the set $K_l = K_l(m) = \{\omega \in H_A : I(\omega) < j_l\}$ does not contain CP except m.

Each of the sets K_l is contained in the following together with a certain neighbourhood and $\bigcup K_l = K^*$. Here, the inequality $I(\omega^l) < I(\omega) \leqslant I(\omega^{l+1})$ is satisfied $K_{l+1} \setminus K_l$ since otherwise there would be a minimum or maximum component in $K_{l+1} \setminus K_l$ contrary to the definition of K_{l+1}. By virtue of the continuous differentiability of the functional $\varphi(\omega)$ and the existence of another CP n the upper bound j^* is finite.

We will show that ∂K^* contains CP. We assume the opposite, i.e. $\partial K^* \cap C_I = \varnothing$. Then for a certain $\delta > 0$ we construct the set $K_\delta^* \equiv K_{j^*+\delta}^*$ which contains no CP. In fact, for the point $y \in K_\delta^* \setminus K^*$ such that $\| y - \omega \|_{H_A} < \delta$, $\omega \in \partial K^*$, we have

$$\| I'(y) - I'(\omega) \| \leqslant M \| y - \omega \|_{H_A} \leqslant M\delta, \ M = 1 + \max \| \varphi''(\omega_0) \|.$$

where the maximum is taken over the values $\omega_0 \in \partial K^* \cup (K_\delta^* \setminus K^*)$. Hence, by using the Lemma 2.3, we deduce

$$\| I'(y) \| \geqslant \| I'(\omega) \| - M\delta \geqslant \alpha - M\delta > \alpha/2$$

if $\delta < \alpha/(2M)$. Consequently, a set K_δ^* is constructed that contains no CP different from m, which contradicts the definition of j^*. Therefore, the existence of a CP of the functional I on ∂K^* is proved. Denoting this point by γ we have $I'(\gamma) = 0$, $I(\gamma) = j^*$, $\gamma \in \partial K^*$. We conclude from Corollary 2.1 that the CP γ cannot be a relative maximum point of the functional I. It will be a saddle point on ∂K^* since it can also not be a relative minimum point of I.

Let us assume the opposite, i.e., the existence of a small number $\varepsilon > 0$ such that for $\omega \in S_\varepsilon \equiv \{\omega \in H_A : \| \omega - \gamma \|_{H_A} < \varepsilon\}$ the inequality $I(\omega) > I(\gamma)$ holds if $\omega \neq \gamma$. Then for the points $\omega \in S_\varepsilon \cap K^*$ we obtain the contradictory inequality $I(\omega) < I(\gamma) = j^*$ by construction.

Theorem 2.2. Let m and n be isolated relative minimum points of the functional $I(\omega)$. There are no other minimum points. Then in the non-degenerate case the functional $I(\omega)$ has a saddle point with index -1.

Proof. By virtue of Corollary 2.1 the CP of the functional $I(\omega)$ cannot be a maximum. The minimum equilibrium index in the case when the equilibrium is non-degenerate equals +1. Indeed, if m is a maximum point, then the spectrum of the Fréchet differential $1 - \varphi''(m)$ is positive and the eigenvalues λ_k of the linear completely continuous with the operator $\varphi''(m)$ satisfy the inequality $\lambda_k < 1$. Then from the Leray-Schauder theorem (/9/, p.141), we obtain that the index of the non-degenerate minimum equals +1.

Furthermore, the rotation of a completely continuous vector field $\omega - \varphi'(\omega)$ on large spheres $S_\rho = \{\omega : \| \omega \|_{H_A} = \rho\}$ equals +1 since this field is homotopic to the field ω (/23/, p.152). Now the theorem follows from the Leray-Schauder principle.

Definition of a well /7, 8/. A connected set in H_A that contains m and consists of those points ω for which $I(\omega) < j^*$ where j^* is the upper bound of those energy values j for which the sets $\{\omega \in H_A : I(\omega) < j\}$ contain no stable equilibria different from m is called the well $J(m)$ (Fig.1a).

Obviously $J(m) = K_{j^*}(m)$. Therefore, by using Theorems 2.1 and 2.2, we obtain for systems

(1.1) with potential energy of the form of the square of the norm plus a weakly-continuous functional /9/ that the existence of saddles with negative index follows from the ambiguity of the stable equilibrium. Here at least one saddle point is found on the boundary of a well of each stable equilibrium. The fact mentioned enables us to estimate the stable equilibrium well depth. Following A.D. Myshkis /7/ we understand the quantity

$$Z(m) = I(\gamma) - I(m)$$

where γ is any unstable equilibrium on $\partial J(m)$, to be the depth of the well $J(m)$ or the equilibrium stability factor.

3. Dynamic snap-through (DS). Astatic Critical load (CL). Many problems on the DS of non-linear elastic systems of the form (1.1) are included in the following general scheme.

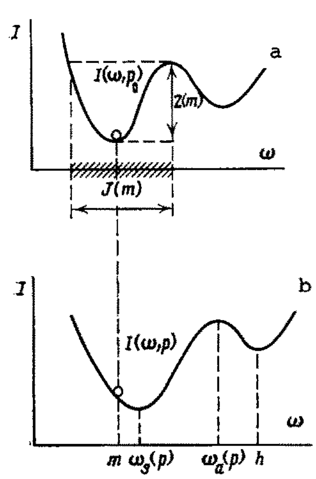

Fig.1

Let $I = I(\omega, p)$ depend on the load parameter p and $\omega_s(p)$ be a continuous single-parameter family of stable equilibria of the system (1.1) corresponding to load parameter values from $[p_0, p_k)$. Let the system be in the equilibrium $\omega_0 = \omega_s(p_0)$ for the value p_0, and then the load parameter changes by a jump and takes the value $p^0 \in [p_0, p_k)$ (to be specific, the case $p^0 > p_0$ is considered, and the case $p^0 < p_0$ can be examined analogously).

The question is whether the system remains in the equilibrium well $\omega_s(p^0)$ or departs from it as time lapses. The situation therefore reduces to an investigation of the behaviour of the solution of the Cauchy problem for system (1.1) with initial data $\omega(0) = \omega_0$, $\omega_t(0) = 0$. If $\omega(t) \in J(\omega_s(p^0))$ for all $t > 0$, then we will say that there is no DS. If $\omega(t)$ turns out to be outside the well $J(\omega_s(p^0))$ for a certain t_* then DS occurs.

We will determine the CL of the DS p_d^* for the fundamental equilibrium $\omega_0 = \omega_s(p_0)$ by setting it equal to the upper bound of the values of p for which the motion of system (1.1) with initial conditions $\omega(0) = \omega_0$, $\omega_t(0) = 0$ remains in the well $J(\omega_s(p))$ for all $t > 0$ (Fig.1b). Therefore, when p slightly exceeds p_d^* at a certain time the motion mentioned will emerge from the equilibrium well $\omega_s(p)$, according to the definition, this will indeed be a DS.

For $p = p^0$ let the domain of possible motions $V(p^0)$ (see (1.6)) satisfy the condition $V(p^0) \subseteq J(\omega(p^0))$. Then there are no DS. We will increase p starting from p^0. By virtue of Theorem 2.1 the DS becomes possible for the least value of p for which the saddle point at $\partial J(\omega_s(p))$ is incident into the domain $V(p)$ defined by the inequality (1.6). There is obviously no DS if $\omega_s(p)$ is the single equilibrium of the system (1.1).

Evaluation of p_d^* involves integration of the non-stationary system (1.1) for different p in a time segment undetermined in advance (and possibly even tending to ∞ as $p \to p_d^*$), which produces serious difficulties. Its main disadvantage is the limiting condition $\omega_t(0) = 0$.

Furthermore, we introduce the astatic CL p_a of the equilibrium ω_0 as the least solution of the equation

$$I(\omega_a(p_a),\ p_a) = I(\omega_0,\ p_a),\quad \omega_0 = \omega_s(p_0) \tag{3.1}$$

where $\omega_a(p_a)$ is the unstable equilibrium at $\partial J(\omega_s(p_a))$ such that $I'(\omega_a(p_a), p_a) = 0$.

Therefore, for $p = p_a$ (and p slightly exceeding p_a) as small as desired, but a directional thrust of the geodesic connecting ω_0 with $\omega_a(p_a)$ in a suitable manner ejects the system from $J(\omega_u(p_a))$.

Let us consider the stationary problem

$$I'(\omega, p) = 0 \tag{3.2}$$

Let p_u denote the upper CL for the buckling of a fundamental family of subcritical equilibria $\omega_s(p)$ and p_l the lower CL of the system, i.e., the least load prior to which the solution of system (3.2) is unique. We assume that $p \in [0, p_u]$, $p_0 = 0$, $\omega_s(0) = 0$, $p_l > 0$, $I(\omega_0, p_a) = 0$. Then if the inequalities

$$I(\omega_u, p_u) < 0,\quad I(\omega_l, p_l) > 0$$

are simultaneously satisfied, where $\omega_u = \lim \omega_s(p)$ as $p \to p_u$, and ω_l is the solution of system (3.2) for $p = p_l$, where $\omega_l \neq \omega_s(p_l)$, then the value p_a satisfies the inequality $p_l < p_a < p_u$.

It is clear that $p_a \leqslant p_d^*$ so that the astatic load yields a lower bound for the dynamic load. It can be assumed that from the practical viewpoint it is more informative since it is referred to more general initial conditions (only the energetic smallness of the quantity $\omega_t(0)$ is required in place of the condition $\omega_t(0) = 0$). Meanwhile, it is considerably easier to calculate the quantity p_a in many cases since only stationary problems need be considered for this according to the theorem. In a number of cases $p_a \approx p_d^*$. Agreement holds if ω_0 lies on the unstable arm of the saddle ω_a.

We consider the set of values of p: $B_p = (p\colon I(y, p) = I(\omega_0, p))$, where y is any solution of the equation $I'(y, p) = 0$ corresponding to unstable equilibrium. It is ordinary possible to arrange the values of $p \in B_p$ in the form of a non-decreasing sequence of numbers p_i $(i = 1, \ldots, k)$. A saddle point y_i corresponds to each p_i. Evidently $p_a \in B_p$ and $p_1 \leqslant p_a$. If the set B_p consists of one point then $p_a = p_1$. If the set B_p consists of more than one point, then a difficulty arises in isolating those saddle points y_i that belong to the boundary of the well $\partial J(\omega_s(p))$. It is hence sufficient to consider only $p_i \in [p_0, p_u)$. Only the solution of the non-stationary problem (1.1) yields a guarantee that the saddle point belongs to $\partial J(\omega_s(p))$.

We will consider the special case when (1.1) describes the vibrations of a mechanical system with N degrees of freedom. Then $H_A = R^N$ is a Euclidean space $\omega(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_N(t)) \in R^N$ are coordinates of the system location at the time t. The potential energy $I(\omega, p)$ is a fairly smooth function in a certain domain $L = \Omega \times [p_0, p_k]$ and growing in $\omega: \forall R > 0 \exists$, there is a $r > 0$ such that $I(\omega, p) > R$ follows from the condition $\|\omega\| > r$ for all $p \in [p_0, p_k]$.

The domain Ω is called the configuration space of the system. Its phase space $M = \Omega \times R^N$ is the space of all possible pairs $(\omega, \omega_t)\colon \omega \in \Omega$, $\omega_t \in R^N$. Under the assumptions made, the Cauchy problem with initial conditions $\omega|_{t=0} = g_0$, $\omega_t|_{t=0} = g_1$ for (1.1) when $(g_0, g_1) \in M$ has a solution (unique because of the smoothness), defined for all $t > 0$. Indeed, because of the growth of the function I in ω, the a priori estimate of the solution for all $t > 0$ follows from the energy dissipation Eq.(1.5).

The critical points of the functional I, the equilibrium points of the system (1.1), satisfy (2.1). For $\beta = 0$ each isolated point of the minimum m is a stable equilibrium of the system (the Lagrange theorem), while for $\beta > 0$ it becomes asympotically stable. The definition of a well and the proof of the theorem about the existence of a saddle point on its boundary in the case of a finite-dimensional space is contained in /7/. The type of saddle point s is governed by the number of inertias of the quadratic form corresponding to the Fréchet differential $1 - \varphi''(s)$.

4. Dynamic snap-through of elastic shells. Let D be a simply-connected domain in the (x, y) plane, bounded by the sufficiently smooth contour $\Gamma = \Gamma_1 \cup \Gamma_2$; ρ, $\varkappa$, s_0 are the internal normal, curvature, and arclength of the curve Γ.

Following /10-12/, we introduce the Hilbert spaces of the functions

1) the space $H = L_2(D)$ with the scalar product

$$(u, v)_H = \int_D uv\,dx\,dy \quad \forall u,\ v \in H$$

2) the space H_1 which is the closure of the set of functions infinitely differentiable in D and satisfying the boundary conditions

$$w|_\Gamma = 0,\quad w_\rho|_{\Gamma_1} = 0 \qquad (4.1)$$
$$[w_{\rho\rho} - \nu\varkappa w_\rho]_{\Gamma_2} = 0,\quad 0 < \nu < {}^1/_2$$

with a finite norm generated by the scalar product

$$(w_1, w_2)_{H_1} = (\Delta^2 w_1, w_2)_H = \qquad (4.2)$$
$$\int_D \{\Delta w_1 \cdot \Delta w_2 - (1-\nu)[w_1, w_2]\}\,dx\,dy$$
$$\Delta w = w_{xx} + w_{yy},\quad \Delta^2 = \Delta\Delta$$
$$[u, v] = u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}$$

H_1 is the energetic space of the biharmonic operator with boundary conditions (4.1) on Γ.

3) in the special case when $\Gamma \equiv \Gamma_1$, we denote the space H_1 by H_2.

We consider a shallow elastic shell with middle surface $z = z(x, y)$, $(x, y) \in D$. For simplicity we consider the shell edge to coincide with the curve Γ and to be stressfree. At the time $t = 0$ a load $q(x, y, p)$ which remains constant for all $t \geqslant 0$ is applied instantaneously to the shell(p is a load parameter and $q(x, y, 0) = 0$).

The equation for the non-linear vibrations of such a shell can be written in the form

$$mhw_{tt} + I'(w, p) = 0,\ I'(w) = \operatorname{grad}_H I(w) \qquad (4.3)$$
$$I(w, p) = \frac{D}{2} I_1(w) + \frac{1}{2Eh} I_2(F) - \int_D qw\,dx\,dy$$
$$I_i(u) = \int_D \{(\Delta u)^2 - (1-\nu_i)[u, u]\}\,dx\,dy \quad (i = 1,\ 2)$$
$$D = Eh^3/[12(1-\nu^2)],\ \nu_1 = \nu,\ \nu_2 = -\nu$$

Here $w(x, y, t)$ is the deflection function that satisfies the boundary conditions (4.1) on Γ, I is the shell potential energy, F is the Airy stress function, m is the mass per unit volume of the shell, ν is Poisson's ratio, h is the shell thickness, and E is Young's modulus. Here $F \in H_2$ and for a given function $w \in H_1$ is determined uniquely by the requirement that the integral identity

$$1/Eh(F, \chi_1)_{H_2} = \int_D \{[z, w] - {}^1/_2[w, w]\}\chi_1\,dx\,dy \tag{4.4}$$

be satisfied for all $\chi_1 \in H_2$.

We will show that the potential energy I satisfies the conditions of Sect.1. According to (4.4), we have

$$\begin{aligned} &1/EhF = F_1 + F_2 \\ &(F_1, \chi_1)_{H_2} = \int_D [z, w]\chi_1\,dx\,dy \\ &(F_2, \chi_1)_{H_2} = -\frac{1}{2}\int_D [w, w]\chi_1\,dx\,dy, \quad \forall\chi_1 \in H_2 \end{aligned} \tag{4.5}$$

Applying (4.5), we deduce from (4.3) /12/

$$\begin{aligned} &1/EhI_2(F) = \Phi_2^* + \Phi_3^* + \Phi_4^* \\ &\Phi_2^* = {}^1/_2\|F\|^2_{H_2} \\ &\Phi_3^* = (F_1, F_2)_{H_2}, \quad \Phi_4^* = {}^1/_2\|F_2\|^2_{H_2} \end{aligned} \tag{4.6}$$

where Φ_i^* are homogeneous functionals of order i relative to w.

Lemma 3.1. /10-12/*. The functionals Φ_i^* $(i = 2, 3, 4)$ and $\int_D qwdxdy$ are weakly continuous in H_1. By using the lemma, we obtain that the functional I can be represented in the form of a sum of $\|w\|_{H_1}^2$ and a weakly continuous functional, i.e., in the form of (1.3). It follows from (4.3) that $I(w)$ is a functional that grows in w (coercivity property). We note that analogous deductions are obtained from the I.I. Vorovich results in the case of boundary conditions corresponding to fixed clamping of the edge.

Therefore, for system (4.3) it is possible to apply the investigations of Sect.3, to introduce the astatic CL and by using the analysis of the non-linear equilibrium equations, to find the lower bound of the critical load for the shell DS.

For example, we will consider the problem on an axisymmetric DS of a spherical shell, clamped freely along the edge and in equilibrium $w_0 = w_s(p_0)$ under the effect of a uniformly distributed external pressure $q = p_0 q_0$ (here $q_0 = 32\,Eh_0^3h\Lambda^{-2}a^{-4}$, $\Lambda^2 = 4\,[3(1-\nu^2)]^{1/2}h_0h^{-1}$, h_0 is the shell rise, h is its thickness, a is the support radius, E is Young's modulus and ν is Poisson's ratio). At the time $t = 0$ the additional pressure $q = (p - p_0)\,q_0$ is applied instantaneously to the shell so that for $t > 0$ a uniformly distributed pressure $q = pq_0$ $(p > p_0)$ is constantly applied to the shell.

In the case of axisymmetric deformation, the shell potential energy is written in dimensionless variables in the form

$$\begin{aligned} &I_0(w, p) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|\Phi\|^2 + 2p\int_0^\Lambda x^3u\,dx \\ &\Phi = \Phi(u) = \int_x^\Lambda t^{-3}\,dt\int_0^t \eta\left(\frac{1}{2}u^2 + \eta u\right)d\eta \\ &\|w\|^2_{H_A} = \|u\|^2 = \int_0^\Lambda \left(xu_x^2 + \frac{u^2}{x}\right)dx, \quad u = w_x \\ &u(0) = u(\Lambda) = w(\Lambda) = 0, \quad I = \frac{h_0^2h^3E}{3(1-\nu^2)\Lambda^2a^2}I_0 \end{aligned} \tag{4.7}$$

The relationship between the dimensionless and dimensional variables is governed by formulas (8) in /3/, $\nu = {}^1/_3$.

To determine the equilibrium astatic CL of w_0 it is necessary to find the value $p \in [p_0, p_u)$ corresponding to points of intersection of the graphs $I_0(w^*, p)$ and $I_0(w_0, p)$ where $w^*(x, p)$ is the solution of the stationary problems $I_0'(w^*, p) = 0$. Only those values of p are selected here that correspond to unstable equilibria. The graphs mentioned are constructed by the alignment method /24, 25/.

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*See also Vorovich, I.I. Certain Mathematical Problems of Non-linear Shell Theory. Doctoral Dissertation. Leningrad State University. Leningrad, 1958.

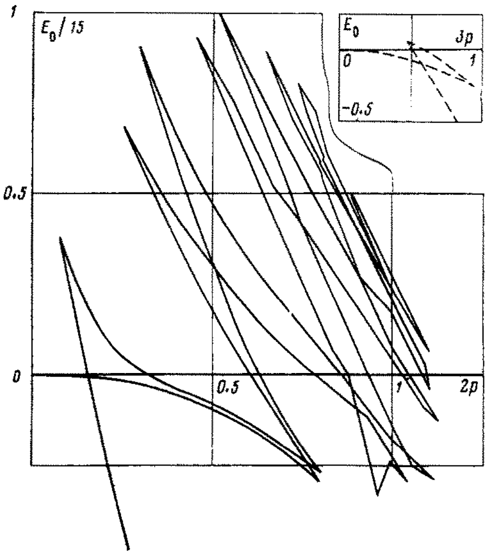

Fig.2

Let $p_0 = 0$. Then $w_0 = w_s(0) = 0$ and $I_0(w_0, p_a) = 0$. To determine the points of the set B_p, we have the equation $I_0(y, p) = 0$, where y is any solution of the equation $I_0'(y, p) = 0$ corresponding to unstable equilibrium. A graph of $E_0 = 10^{-2} I_0(w^*, p)$ for $\Lambda = 6$ is represented in the right upper corner of Fig.2. The astatic CL p_a of the equilibrium $w_0 = w_s(0) = 0$ is the point of intersection of the p axis with the unstable equilibrium branch of the graph of E_0, i.e., $p_a = 0.190$. We note that because of a direct numerical integration of problem (4.3), (4.7) using an implicit finite-difference scheme and application of the Budiansky-Roth criterion /2, 3/, we obtain $p_d = 0.195$ for the critical DS value of the equilibrium $w_0 = 0$. We here have for the initial conditions $[w = w_t]_{t=0} = 0$. Analogously, we find $p_a = 0.190$, $p_d = 0.192$ for $\Lambda = 5.5$.

For $\Lambda = 12$ the graph of E_0 is represented by the solid lines in Fig.2. In this case $p_d = 0.29$. To determine the load p_a of the equilibrium $w_0 = 0$ we examine the set B_p introduced in Sect.3, which consists of the points $p_1 = 0.159$, $p_2 = 0.303$, $p_3 = 0.427$, $p_4 = 0.438$, $p_5 = 0.519$, $p_6 = 0.546$. Furthermore, we have $p_u = 0.396$, and only the points p_1 and p_2 belong to the interval $[0, p_u)$. The point p_2 is found on the graph of $E_0(p)$ on the unstable branch of solutions having a common point p_u with the stable pre-critical equilibrium branch, while the point p_1 is on the unstable branch having the common point $p_l = 0.039$ with the branch of the stable equilibrium family deep in the post-critical stage. We hence conclude that $p_a = 0.303$; p_d is obtained by integrating with respect to t between 0 and 750. We note that the values of p_a and the CL p_d for spherical and conical shells under the fundamental kinds of boundary conditions and different values of Λ are presented in /15, 16/.* (*See also, Srubshchik, L.S., On the critical pressure of dynamic snap-through of elastic spherical and conical shells. Rostov-on-Don, 1983. Deposited in VINITI 24-03-83, No.1506-83; and the correction to /15/ printed in the Dokl. Akad. Nauk SSSR, Vol.282, No.2, 264, 1985).

There results from theorems on the uniqueness of the solution and the discussion in Sect. 3, that axisymmetric DS is impossible for circular plates and spherical shells with a fairly small ratio h_0/h.

The solution of the problem of determining the astatic CL taking non-symmetric deformations into account for a shallow spherical shell is unknown as yet and will become possible when values of the shell potential energy have been evaluated on all the equilibrium paths in the interval (p_l, p_u).

In conclusion, we note that Friedrichs /26/ introduced the intermediate critical load $p_M \in (p_l, p_u)$ to explain the snap-through mechanism of a spherical shell, where the fundamental and buckling equilibrium modes have equal potential energies. Values of p_M have been evaluated in /24/, for instance. The values of p_M are obviously substantially less than the values of p_a.

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